Existence of Periodic Points and Sharkovsky's Theorem

Trevor Hyde

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Existence of Periodic Points

Central Question in Arithmetic Dynamics:
If K is a field and $f(x) \in K[x]$ is a polynomial, for which $n \geq 1$ does $f(x)$ have a periodic point of period n in K?

- The answer really depends on the field K!

Theorem (Baker, 1964)

*If $f(x) \in \mathbb{C}[x]$ has degree at least 2 and $n \geq 1$, then there exists a point $\alpha \in \mathbb{C}$ with primitive period n **unless** $f(x)$ is conjugate to $x^2 - \frac{3}{4}$ and $n = 2$.*

Conjecture (Morton-Silverman, 1994)

Suppose $d \geq 2$, then there exists an absolute bound $B(d)$ such that for any polynomial $f(x) \in \mathbb{Q}[x]$ of degree d, if $\alpha \in \mathbb{Q}$ has primitive period n, then $n \leq B(d)$.

Existence of Periodic Points over $\mathbb{R}$

Consider the following total ordering $\lessdot$ of the positive integers:

$$3 \lessdot 5 \lessdot 7 \lessdot 9 \lessdot \cdots \lessdot 2\cdot 3 \lessdot 2\cdot 5 \lessdot 2\cdot 7 \lessdot \cdots$$
$$\cdots \lessdot 4\cdot 3 \lessdot 4\cdot 5 \lessdot 4\cdot 7 \lessdot \cdots \lessdot 8\cdot 3 \lessdot 8\cdot 5 \lessdot 8\cdot 7 \lessdot \cdots$$
$$\cdots \lessdot 8 \lessdot 4 \lessdot 2 \lessdot 1$$

(Odds ≥ 3) $\lessdot$ ($2 \cdot$ Odds ≥ 3) $\lessdot$ ($4 \cdot$ Odds ≥ 3)
$\lessdot$ ($8 \cdot$ Odds ≥ 3) $\lessdot \cdots\cdots \lessdot$ (Powers of 2 in reverse order)

- This is called the **Sharkovsky ordering**.

Existence of Periodic Points over $\mathbb{R}$

$3 \lessdot 5 \lessdot 7 \lessdot 9 \lessdot \cdots \lessdot 2 \cdot 3 \lessdot 2 \cdot 5 \lessdot 2 \cdot 7 \lessdot \cdots \lessdot 4 \cdot 3 \lessdot 4 \cdot 5 \lessdot 4 \cdot 7 \lessdot \cdots$

Theorem (Sharkovsky, 1960's)

1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a* ***continuous function****. If there exists a point $\alpha \in \mathbb{R}$ with primitive period m, then there exists a point $\beta \in \mathbb{R}$ with primitive period n for every $m \lessdot n$.*

2. *For every positive integer m there exists some continuous function $f_m : \mathbb{R} \to \mathbb{R}$ such that f_m has a point $\alpha \in \mathbb{R}$ of primitive period n if and only if $m \lessdot n$.**

▷ For example, if $f : \mathbb{R} \to \mathbb{R}$ has a point $\alpha \in \mathbb{R}$ with primitive period 3, then f has points $\beta \in \mathbb{R}$ with **every** primitive period!

$\lessdot 8 \cdot 3 \lessdot 8 \cdot 5 \lessdot 8 \cdot 7 \lessdot \cdots \lessdot 16 \cdot 3 \lessdot 16 \cdot 5 \lessdot 16 \cdot 7 \lessdot \cdots \lessdot 8 \lessdot 4 \lessdot 2 \lessdot 1$

Realizing Sharkovsky Tails

How do we find functions f whose real periodic points exactly realize any tail of the Sharkovsky ordering?

▷ Look no further than $x^2 + c$!

Let $\Phi_n(x, c)$ be the n**th dynatomic polynomial** for $x^2 + c$.

▷ $\Phi_n(a, b) = 0$ if and only if a (spiritually) has primitive period n for $x^2 + b$.

$$\Phi_1(x, c) = x^2 - x + c$$
$$\Phi_2(x, c) = x^2 + x + c + 1$$

For $n \geq 1$, let $c_n = \sup\{c' : \Phi_n(x, c') \text{ has a real root.}\}$

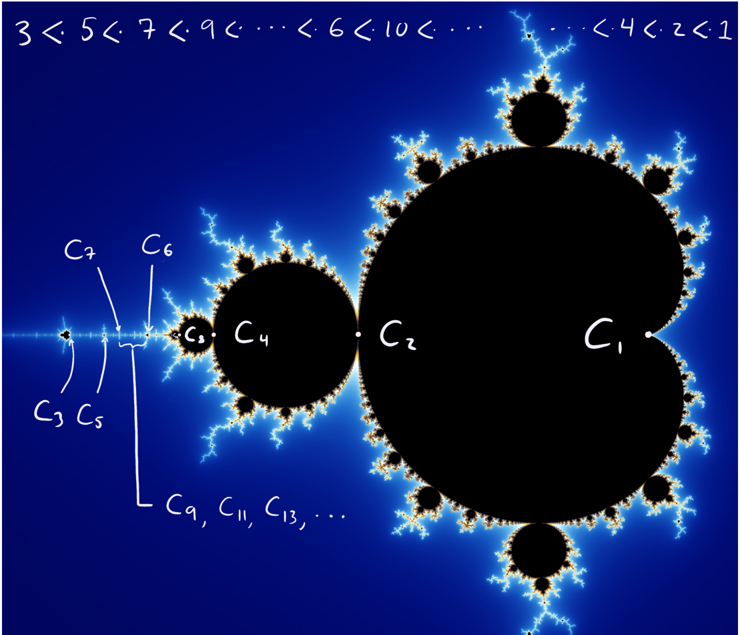
Realizing Sharkovsky Tails

For $n \geq 1$, let $c_n = \sup\{c' : \Phi_n(x, c') \text{ has a real root.}\}$

- If $c > c_m$, then $x^2 + c$ **does not** have a real m-cycle.
- If $c < c_m$, then $x^2 + c$ **does** have a real m-cycle.
- The c_n's exactly line up in the Sharkovsky order!

$$-\frac{7}{4} = c_3 < c_5 < c_7 < \cdots < c_6 < c_{15} < c_{21} < \cdots$$
$$c_{12} < c_{20} < c_{28} < \cdots < c_4 < c_2 < c_1 = \frac{1}{4}$$

Sharkovsky tails and the Mandelbrot set



Sharkovsky's Implication

Theorem (Sharkovsky, 1960's)

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. If there exists a point $\alpha \in \mathbb{R}$ with primitive period m, then there exists a point $\beta \in \mathbb{R}$ with primitive period n for every $m \lessdot n$.

How do you prove this??

- We start with some cycle of period m for a continuous map f and then have to use it to cook up cycles of every length n for $m \lessdot n$.

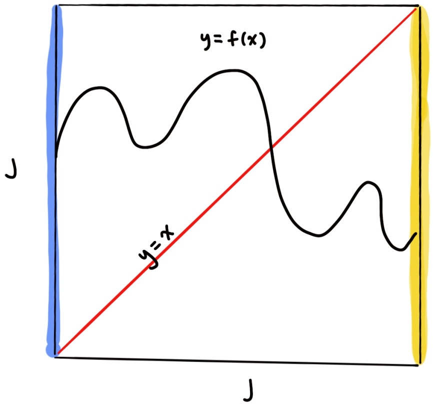
Seems hard.

- We must need a deep, powerful theorem to do this...

Intermediate Value Theorem

Theorem (IVT)

If $J = [a, b]$ is a closed interval and $f : J \to J$ is a continuous function, then there exists a fixed point $p \in J$, that is $f(p) = p$.



Covering Intervals

If I and J are closed intervals, then we say I **covers** J if $J \subseteq f(I)$

- We write $I \xrightarrow{f} J$ or just $I \to J$ if f is understood.

Lemma (Itinerary Lemma)

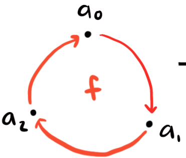
*Suppose $J_0 \xrightarrow{f} J_1 \xrightarrow{f} J_2 \xrightarrow{f} \cdots \xrightarrow{f} J_{n-1} \xrightarrow{f} J_0$ is an n-**loop** of closed intervals. Then there exists a point $p \in J_0$ **following the loop**, which means*

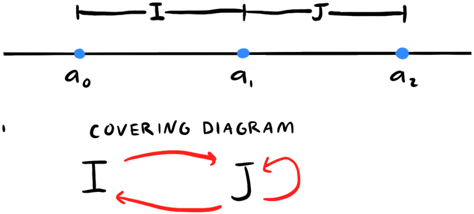
1. $f^k(p) \in J_k$
2. $f^n(p) = p$.

An n-loop of intervals gives us a point p of period n.

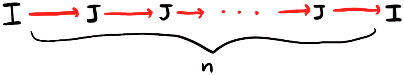
- **Caveat:** We need to check that p has *primitive* period n.

3-Cycles $\Longrightarrow$ All-Cycles





For any $n \geq 2$ we get an n-loop:

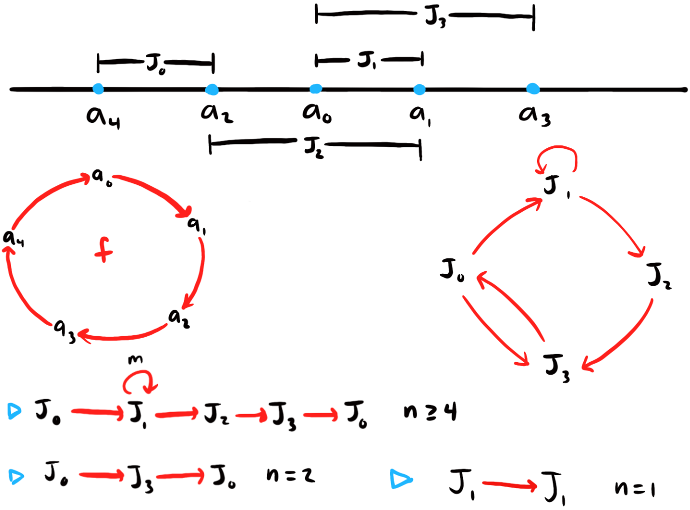


Itin. Lem.: $\exists p \in I$ such that $f^n(p) = p$ and $f^k(p) \in J$ for $k < n$.

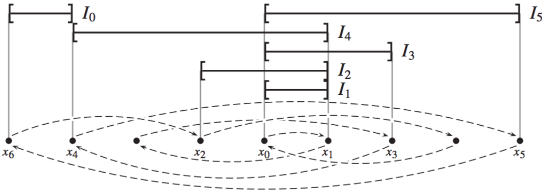
▷ Since I and J only overlap at the endpoints (period 3) p must have primitive period n.

5-Cycles $\Longrightarrow$ Almost All-Cycles

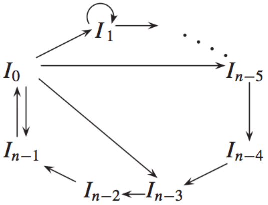
For larger starting cycles we use the same idea, but there are more ways the 5-cycle can be arranged on the line.



The Rest of the Story



Keith Burns and Boris Hasselblatt, The Sharkovsky Theorem: A Natural Direct Proof, The American Mathematical Monthly, 118:3, (2011), 229-244.



Dynamics with Several Functions

Classical dynamics: Study iterations of a single function.

$$f : X \to X$$

New direction: Study mixed iterations of several functions!

$$f_1, f_2, \dots, f_m : X \to X$$

Dynamical system $=$ (Representation of a) Semigroup

$$f : X \to X \iff \langle f \rangle = \{f^n : n \geq 0\} = \textbf{Cyclic semigroup}$$

$$f_1, f_2, \dots, f_m : X \to X \iff \langle f_1, f_2, \dots, f_m \rangle = \text{Semigroup}$$

Dynamical Semigroups

BIG GOAL: Develop the theory of dynamical semigroups.

$$\langle f_1, f_2, \ldots, f_m \rangle$$

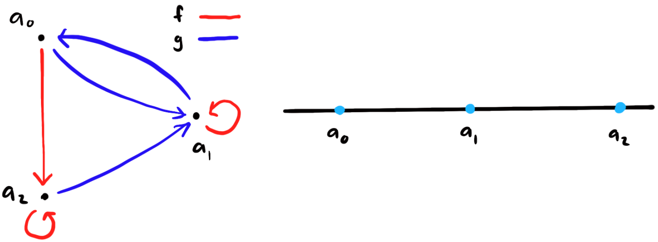
- Which aspects of "cyclic dynamics" generalize?
- What new phenomena arise in a non-cyclic setting?

How does Sharkovsky's theorem generalize to the non-cyclic case?

One Direction

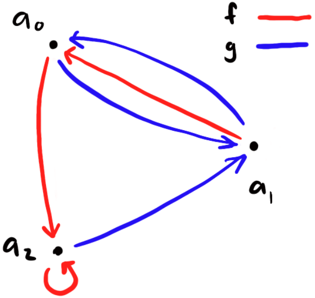
Suppose $f_1, f_2, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ are continuous functions.

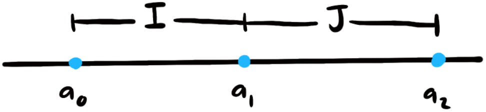
- Let $S \subseteq \mathbb{R}$ be a finite set such that $f_i(S) \subseteq S$ for $1 \leq i \leq m$.
- S is a generalization of a (pre)periodic cycle.

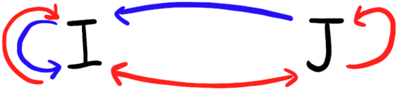


Given that $D = \langle f_1, f_2, \ldots, f_m \rangle$ has a real portrait S of this type, for which $w = f_{i_1} f_{i_2} \cdots f_{i_k} \in D$ does it follow that w has a real fixed point?

Example







$\therefore\ \exists\, p \in \mathbb{R}$ such that $fggfg(p) = p$

Thank you!